

Associating quantum vertex algebras to certain deformed Heisenberg Lie algebras

Haisheng Li¹

Department of Mathematical Sciences
Rutgers University, Camden, NJ 08102

Abstract

We associate quantum vertex algebras and their ϕ -coordinated quasi modules to certain deformed Heisenberg algebras.

1 Introduction

It is now well known (see [FZ]; cf. [LL]) that certain infinite-dimensional Lie algebras such as affine (Kac-Moody) Lie algebras, the Virasoro algebra, and Heisenberg algebras of a certain type can be canonically associated with vertex algebras and their modules. This association can be briefly outlined as follows: First, the canonical generating functions of their generators are *mutually local* in the sense that for any two generating functions $a(x)$ and $b(x)$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1). \quad (1.1)$$

Second, a conceptual result of [Li1] states that for any vector space W , any local subset of $\text{Hom}(W, W((x)))$, alternatively denoted by $\mathcal{E}(W)$, generates a vertex algebra canonically with W as a faithful module. Moreover, the generated vertex algebras have a canonical module structure identified as a so-called vacuum module for the corresponding Lie algebra.

In this paper, we study two deformed Heisenberg Lie algebras, which have appeared in the study of quantum algebras, and our goal is to associate vertex algebras or their likes to these Lie algebras. The first we are concerned about is the Heisenberg algebra (see [FR]) with generators a_n for $n \in \mathbb{Z}$, subject to relations

$$[a_m, a_n] = [m]_q \delta_{m+n,0} \quad (1.2)$$

for $m, n \in \mathbb{Z}$, where the complex parameter q is neither zero nor a root of unity and where $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$, while the second is the Heisenberg algebra (cf. [BP]) with generators b_n for $n \in \mathbb{Z}$, subject to relations

$$[b_m, b_n] = m(1 - q^{|m|}) \delta_{m+n,0}. \quad (1.3)$$

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Heisenberg Lie algebras are the simplest nonabelian Lie algebras, however, as we shall see, associating vertex algebras or their likes to them is by no means straightforward. The main purpose of this paper is to provide a simple but illuminating example.

For the first one, forming a generating function $a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n}$, we have

$$[a(x_1), a(x_2)] = \frac{1}{q - q^{-1}} \left(\delta \left(\frac{qx_2}{x_1} \right) - \delta \left(\frac{x_2}{qx_1} \right) \right), \quad (1.4)$$

while for the second one, using generating function $b(x) = \sum_{n \in \mathbb{Z}} b_n x^{-n}$ we have

$$[b(x_1), b(x_2)] = \left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{x_2}{x_1} \right) + \frac{qx_1/x_2}{(1 - qx_1/x_2)^2} - \frac{qx_2/x_1}{(1 - qx_2/x_1)^2}. \quad (1.5)$$

Notice that neither generating function $a(x)$ nor $b(x)$ forms a local set. Thus the results of [Li1] are *not* applicable to these two Lie algebras.

In the past, for various purposes we have significantly generalized the results of [Li1] in several directions. One of the generalizations was obtained in [Li4] with a purpose to associate vertex algebras to a family of infinite-dimensional Lie algebras called quantum tori Lie algebras (see [G-K-L], [G-K-K]). For these Lie algebras, their generating functions satisfy a generalized locality relation

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1) \quad (1.6)$$

for some nonzero homogeneous polynomial $p(x_1, x_2)$. Motivated by this, we then introduced a notion of quasi locality, using commutativity relation (1.6). It was proved that for any vector space W , every quasi local subset of $\mathcal{E}(W)$ generates a vertex algebra in a certain *new* way, with W as what we called a quasi module. In this way, we obtained a new construction of vertex algebras together with a theory of quasi modules for vertex algebras, not to mention that vertex algebras were associated to quantum tori Lie algebras as an application.

As with the two aforementioned Heisenberg algebras, we see that the generating function $a(x)$ of the first one forms a quasi local set, so that one can use the results of [Li4] to associate a vertex algebra to this Lie algebra. However, the structure of the associated vertex algebra is not as neat as expected. As for the second one, since the generating function $b(x)$ does not give rise to a quasi local set, the construction of [Li4] is not applicable. In fact, the notion of vertex algebra is not general enough; we need a generalization of the notion of vertex algebra.

Inspired by Etingof-Kazhdan's notion of quantum vertex operator algebra (see [EK]), in [Li2] we introduced a notion of weak quantum vertex algebra and a notion of quantum vertex algebra, greatly generalizing the notion of vertex superalgebra. Furthermore, we established a conceptual construction. For a vector space W , a subset U of $\mathcal{E}(W)$ is said to be \mathcal{S} -local if for any $u(x), v(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

such that

$$(x_1 - x_2)^k u(x_1)v(x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1)v^{(i)}(x_2)u^{(i)}(x_1)$$

for some nonnegative integer k . It was proved that any \mathcal{S} -local subset of $\mathcal{E}(W)$ generates a weak quantum vertex algebra with W as a faithful module.

Note that the generating functions of the two aforementioned Heisenberg algebras do *not* give rise to \mathcal{S} -local sets. Thus the results of [Li2] are not applicable *either*. To deal with the two Heisenberg algebras, especially the second one, we shall need a different idea.

In the general field of vertex algebras, a fundamental problem (see [FJ], [EFK]) has been to develop a theory of quantum vertex algebras so that quantum vertex algebras can be canonically associated to quantum affine algebras. For many years, solving this very problem has been the main driving force for us to study quantum vertex algebras. Finally, we made a significant progress in [Li6]. As the crucial step, we developed a theory of what we called ϕ -coordinated quasi modules for weak quantum vertex algebras (actually for more general nonlocal vertex algebras), where $\phi = \phi(x, z) \in \mathbb{C}((x))[[z]]$ is a formal series satisfying

$$\phi(x, 0) = x, \quad \phi(\phi(x, z_1), z_2) = \phi(x, z_1 + z_2).$$

All such ϕ were completely determined therein, and two particular examples are $\phi(x, z) = x + z$ and $\phi(x, z) = xe^z$. For $\phi(x, z) = x + z$, the notion of ϕ -coordinated quasi module reduces to the notion of quasi module. What is important for us to deal with quantum affine algebras is the case with $\phi = xe^z$. In this case, we obtained a general construction. Let W be a general vector space. A subset U of $\mathcal{E}(W)$ is said to be *quasi \mathcal{S}_{trig} -local* if for any $a(x), b(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r),$$

where $\mathbb{C}(x)$ denotes the field of rational functions, such that

$$p(x_1, x_2)u(x_1)v(x_2) = p(x_1, x_2) \sum_{i=1}^r f_i(x_1/x_2)v^{(i)}(x_2)u^{(i)}(x_1)$$

for some nonzero polynomial $p(x_1, x_2)$. It was proved that any quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$ generates a weak quantum vertex algebra in a certain way with W as a ϕ -coordinated quasi module where $\phi(x, z) = xe^z$. As an application, we have successfully associated weak quantum vertex algebras to quantum affine algebras.

In this current paper, by applying the results of [Li6] we associate vertex algebras and ϕ -coordinated quasi modules to the first Lie algebra, whereas we associate quantum vertex algebras to the second. Note that by [Li6] *weak* quantum vertex algebras can be associated to them *conceptually*. The main task here is to construct

the desired weak quantum vertex algebras concretely and show that they are indeed quantum vertex algebras. To explicitly determine the associated vertex algebras and quantum vertex algebras, we introduce and employ certain Heisenberg algebras. We show that the associated quantum vertex algebras are simple. This as an illustrating example shows how one can associate quantum vertex algebras to more general infinite-dimensional Lie algebras.

This paper is organized as follows: In Section 2, we recall some necessary notions and results. In Section 3, we study the first deformed Heisenberg algebra in the context of vertex algebras and their ϕ -coordinated quasi modules. In Section 4, we study the second deformed Heisenberg algebra in terms of quantum vertex algebras and their ϕ -coordinated quasi modules.

2 Quantum vertex algebras and their ϕ -coordinated quasi modules

This is a preliminary section. As we need, we recall the basic notions and results, including the notions of (weak) quantum vertex algebra and ϕ -coordinated quasi module, and also including the conceptual constructions.

We begin by recalling from [Li2] the notion of weak quantum vertex algebra, which generalizes the notion of vertex algebra and that of vertex superalgebra.

Definition 2.1. A *weak quantum vertex algebra* is a vector space V equipped with a linear map

$$Y(\cdot, x) : V \rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]]$$

and a vector $\mathbf{1} \in V$, called the *vacuum vector*, satisfying the following conditions: For $v \in V$,

$$Y(\mathbf{1}, x)v = v, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v;$$

For $u, v, w \in V$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w; \quad (2.1)$$

and for $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}((x)) \quad \text{for } i = 1, \dots, r$$

such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1)Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \quad (2.2)$$

for some nonnegative integer k .

A *rational quantum Yang-Baxter operator* on a vector space U is a linear operator

$$\mathcal{S}(x) : U \otimes U \rightarrow U \otimes U \otimes \mathbb{C}((x))$$

satisfying the quantum Yang-Baxter equation

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z) = \mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x).$$

It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x) = 1,$$

where $\mathcal{S}^{21}(x) = \sigma\mathcal{S}(x)\sigma$ with σ denoting the flip operator on $U \otimes U$.

Definition 2.2. A *quantum vertex algebra* is a weak quantum vertex algebra V equipped with a unitary rational quantum Yang-Baxter operator $\mathcal{S}(x)$ on V , satisfying

$$\mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v \quad \text{for } v \in V, \quad (2.3)$$

$$[\mathcal{D} \otimes 1, \mathcal{S}(x)] = -\frac{d}{dx}\mathcal{S}(x), \quad (2.4)$$

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V, \quad (2.5)$$

$$\mathcal{S}(x_1)(Y(x_2) \otimes 1) = (Y(x_2) \otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1 + x_2). \quad (2.6)$$

We denote a quantum vertex algebra by a pair (V, \mathcal{S}) .

Note that this very notion is a slight modification of the same named notion in [Li2] and [Li3] with extra axioms (2.3) and (2.6).

As we need, we recall the following important notion due to Etingof and Kazhdan (see [EK]):

Definition 2.3. A weak quantum vertex algebra V is said to be *non-degenerate* if for every positive integer n , the linear map

$$Z_n : V^{\otimes n} \otimes \mathbb{C}((x_1)) \cdots ((x_n)) \rightarrow V((x_1)) \cdots ((x_n)),$$

defined by

$$Z_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f) = fY(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)\mathbf{1}$$

for $v^{(1)}, \dots, v^{(n)} \in V$, $f \in \mathbb{C}((x_1)) \cdots ((x_n))$, is injective.

The following result can be found in [Li2] (cf. [EK]):

Proposition 2.4. *Let V be a weak quantum vertex algebra. Assume that V is non-degenerate. Then there exists a linear map $\mathcal{S}(x) : V \otimes V \rightarrow V \otimes V \otimes \mathbb{C}((x))$, which is uniquely determined by*

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V.$$

Furthermore, (V, \mathcal{S}) carries the structure of a quantum vertex algebra and the following relation holds

$$[1 \otimes \mathcal{D}, \mathcal{S}(x)] = \frac{d}{dx}\mathcal{S}(x). \quad (2.7)$$

Remark 2.5. Note that a quantum vertex algebra was defined as a pair (V, \mathcal{S}) . In view of Proposition 2.4, the term “a non-degenerate quantum vertex algebra” (without reference to a quantum Yang-Baxter operator) is unambiguous. Furthermore, if V is of countable dimension over \mathbb{C} and if V as a V -module is irreducible, then by Corollary 3.10 of [Li3], V is non-degenerate. In view of this, the term “irreducible quantum vertex algebra” is also unambiguous.

We recall from [Li2] a conceptual construction of weak quantum vertex algebras and their modules. Let W be a general vector space. Set

$$\mathcal{E}(W) = \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]]. \quad (2.8)$$

The identity operator on W , denoted by 1_W , is a typical element of $\mathcal{E}(W)$.

Definition 2.6. A subset U of $\mathcal{E}(W)$ is said to be \mathcal{S} -local if for any $a(x), b(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}((x)) \quad \text{for } i = 1, \dots, r$$

such that

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1) u^{(i)}(x_2) v^{(i)}(x_1) \quad (2.9)$$

for some nonnegative integer k , where by convention $f_i(x_2 - x_1)$ is understood as an element of $\mathbb{C}((x_2))((x_1))$.

Let U be an \mathcal{S} -local subset of $\mathcal{E}(W)$ and let $a(x), b(x) \in U$. Notice that (2.9) implies

$$(x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (2.10)$$

Define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}(a(x), z) b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}(a(x), z) b(x) = z^{-k} \left((x_1 - x)^k a(x_1) b(x) \right) \Big|_{x_1=x+z}, \quad (2.11)$$

where k is any nonnegative integer such that (2.10) holds. It was shown that

$$\begin{aligned} & a(x)_n b(x) \\ &= \text{Res}_{x_1} \left((x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n \sum_{i=1}^r f_i(x - x_1) u^{(i)}(x) v^{(i)}(x_1) \right), \end{aligned}$$

where $u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$ such that (2.9) holds.

An \mathcal{S} -local subspace K of $\mathcal{E}(W)$ is said to be $Y_{\mathcal{E}}$ -closed if

$$a(x)_n b(x) \in K \quad \text{for } a(x), b(x) \in K, \quad n \in \mathbb{Z}.$$

The following result was obtained in [Li2] (Theorem 5.8):

Theorem 2.7. *Let U be an \mathcal{S} -local subset of $\mathcal{E}(W)$. Then there exists a $Y_{\mathcal{E}}$ -closed \mathcal{S} -local subspace of $\mathcal{E}(W)$, containing 1_W and U . Denote by $\langle U \rangle$ the smallest such \mathcal{S} -local subspace. Then $(\langle U \rangle, Y_{\mathcal{E}}, 1_W)$ carries the structure of a weak quantum vertex algebra and W is a faithful $\langle U \rangle$ -module with $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle$.*

Throughout this section, we let

$$\phi = \phi(x, z) = xe^z \in \mathbb{C}[[x, z]].$$

The following notion was introduced in [Li6]:

Definition 2.8. Let V be a weak quantum vertex algebra. A ϕ -coordinated quasi V -module is a vector space W equipped with a linear map

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]],$$

satisfying the conditions that

$$Y_W(\mathbf{1}, x) = 1_W \quad (\text{the identity operator on } W)$$

and that for any $u, v \in V$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (2.12)$$

and

$$p(x_2e^{x_0}, x_2)Y_W(Y(u, x_0)v, x_2) = (p(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=x_2e^{x_0}}. \quad (2.13)$$

As we need, next we recall the conceptual construction from [Li6]. Let W be a general vector space.

Definition 2.9. A subset U of $\mathcal{E}(W)$ is said to be *quasi local* (see [Li4]) if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $p(x_1, x_2)$ such that

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1). \quad (2.14)$$

A subset U is said to be *quasi S_{trig} -local* if for any $a(x), b(x) \in U$, there exist

$$u^{(i)}(x), v^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}(x) \quad \text{for } i = 1, \dots, r$$

such that

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2) \sum_{i=1}^r \iota_{x_2, x_1}(f_i(x_1/x_2))u^{(i)}(x_2)v^{(i)}(x_1) \quad (2.15)$$

for some nonzero polynomial $p(x_1, x_2)$, where $\mathbb{C}(x)$ denotes the field of rational functions and ι_{x_2, x_1} is the canonical field embedding of $\mathbb{C}(x_1, x_2)$ into $\mathbb{C}((x_2))((x_1))$.

Let U be a quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$ and let $a(x), b(x) \in U$. Notice that (2.15) implies

$$p(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (2.16)$$

Define $a(x)_n^e b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n^e b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = \iota_{x,z}(1/p(xe^z, x)) (p(x_1, x)a(x_1)b(x))|_{x_1=xe^z}. \quad (2.17)$$

A quasi \mathcal{S}_{trig} -local subspace K of $\mathcal{E}(W)$ is said to be $Y_{\mathcal{E}}^e$ -closed if

$$a(x)_n^e b(x) \in K \quad \text{for } a(x), b(x) \in K, \quad n \in \mathbb{Z}.$$

The following was proved in [Li6]:

Proposition 2.10. *Let V be a $Y_{\mathcal{E}}^e$ -closed quasi compatible subspace of $\mathcal{E}(W)$. Suppose*

$$a(x), b(x), u_i(x), v_i(x) \in V, \quad 0 \neq p(x) \in \mathbb{C}[x], \quad q_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

satisfy

$$p(x_1/x_2)a(x_1)b(x_2) = \sum_{i=1}^r p(x_1/x_2)\iota_{x_2, x_1}(q_i(x_1/x_2))u_i(x_2)v_i(x_1). \quad (2.18)$$

Then

$$\begin{aligned} & p(e^{x_1-x_2})Y_{\mathcal{E}}^e(a(x), x_1)Y_{\mathcal{E}}^e(b(x), x_2) \\ &= p(e^{x_1-x_2}) \sum_{i=1}^r \iota_{x_2, x_1}(q_i(e^{x_1-x_2}))Y_{\mathcal{E}}^e(u_i(x), x_2)Y_{\mathcal{E}}^e(v_i(x), x_1). \end{aligned} \quad (2.19)$$

Furthermore, we have

$$\begin{aligned} & (x_1 - x_2)^k Y_{\mathcal{E}}^e(a(x), x_1)Y_{\mathcal{E}}^e(b(x), x_2) \\ &= (x_1 - x_2)^k \sum_{i=1}^r \iota_{x_2, x_1}(q_i(e^{x_1-x_2}))Y_{\mathcal{E}}^e(u_i(x), x_2)Y_{\mathcal{E}}^e(v_i(x), x_1), \end{aligned} \quad (2.20)$$

where k is the multiplicity of zero of $p(x)$ at $x = 1$.

Theorem 2.11. *Let U be a quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$. There exists a $Y_{\mathcal{E}}^e$ -closed quasi \mathcal{S}_{trig} -local subspace containing U and 1_W . Denote by $\langle U \rangle_e$ the smallest such subspace. Then $(\langle U \rangle_e, Y_{\mathcal{E}}^e, 1_W)$ carries the structure of a weak quantum vertex algebra and W is a ϕ -coordinated quasi $\langle U \rangle_e$ -module where $Y_W(\cdot, x)$ is given by $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle_e$.*

Combining Theorem 2.11 with Proposition 2.10 we immediately have:

Corollary 2.12. *Let U be a quasi local subset of $\mathcal{E}(W)$. Then the weak quantum vertex algebra $\langle U \rangle_e$ obtained in Theorem 2.11 is a vertex algebra.*

Definition 2.13. Let V be a weak quantum vertex algebra, G an automorphism group of V , and $\chi : G \rightarrow \mathbb{C}^\times$ a group homomorphism. We say a ϕ -coordinated quasi V -module (W, Y_W) is (G, χ) -covariant if

$$Y_W(gu, x) = Y_W(u, \chi(g)x) \quad \text{for } g \in G, u \in V. \quad (2.21)$$

In case that G is a subgroup of \mathbb{C}^\times with χ the embedding, we simply drop χ from the notion.

The following is a useful technical result:

Lemma 2.14. *Let V , G , and χ be given as in Definition 2.13. Assume that (W, Y_W) is a ϕ -coordinated quasi V -module and that U is a G -submodule of V which generates V such that (2.21) holds for $g \in G$, $u \in U$. Then (W, Y_W) is a (G, χ) -covariant ϕ -coordinated quasi V -module.*

Proof. Suppose that $u, v \in V$ satisfy

$$Y_W(gu, x) = Y_W(u, \chi(g)x), \quad Y_W(gv, x) = Y_W(v, \chi(g)x) \quad \text{for } g \in G.$$

Note that

$$Y_W(gY(u, z)v, x) = Y_W(Y(gu, z)gv, x).$$

There exists a nonzero homogeneous polynomial $p(x_1, x_2)$ such that

$$p(xe^z, x)Y_W(Y(gu, z)gv, x) = (p(x_1, x)Y_W(gu, x_1)Y_W(gv, x))|_{x_1=x e^z}.$$

Then

$$\begin{aligned} & (p(x_1, x)Y_W(gu, x_1)Y_W(gv, x))|_{x_1=x e^z} \\ &= (p(x_1, x)Y_W(u, \chi(g)x_1)Y_W(v, \chi(g)x))|_{x_1=x e^z} \\ &= (p(\chi(g)^{-1}x_1, x)Y_W(u, x_1)Y_W(v, \chi(g)x))|_{x_1=\chi(g)x e^z} \\ &= p(xe^z, x)Y_W(Y(u, z)v, \chi(g)x). \end{aligned}$$

Consequently, we get

$$p(xe^z, x)Y_W(Y(gu, z)gv, x) = p(xe^z, x)Y_W(Y(u, z)v, \chi(g)x).$$

It was proved in [Li6] that $p(xe^z, z)$ is a nonzero element of $\mathbb{C}[[x, z]] \subset \mathbb{C}((x))((z))$. Multiplying both sides by the inverse of $p(xe^z, x)$ in $\mathbb{C}((x))[[z]]$ we obtain

$$Y_W(Y(gu, z)gv, x) = Y_W(Y(u, z)v, \chi(g)x).$$

Since U generates V , it now follows that (2.21) holds for $g \in G$, $u \in V$. \square

3 q -Heisenberg Lie algebras

In this section we study a certain q -Heisenberg Lie algebra in terms of vertex algebras. We explicitly construct some vertex algebras and then associate these vertex algebras and their ϕ -coordinated quasi modules to the Lie algebra.

We consider the following q -analog of the standard Heisenberg Lie algebra.

Definition 3.1. Let q be a nonzero complex number with $q \neq \pm 1$. Denote by H_q the Lie algebra with generators c and β_m ($m \in \mathbb{Z}$), where c is central, subject to relations

$$[\beta_m, \beta_n] = [m]_q \delta_{m+n,0} c, \quad (3.1)$$

where $[m]_q$ is the q -integer defined by

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Form the generating function

$$\beta(x) = \sum_{n \in \mathbb{Z}} \beta_n x^{-n}. \quad (3.2)$$

The defining relations (3.1) are equivalent to

$$[\beta(x_1), \beta(x_2)] = \frac{1}{q - q^{-1}} \left(\delta \left(\frac{qx_2}{x_1} \right) - \delta \left(\frac{x_2}{qx_1} \right) \right) c. \quad (3.3)$$

From this we get

$$(x_1 - qx_2)(qx_1 - x_2)[\beta(x_1), \beta(x_2)] = 0. \quad (3.4)$$

Remark 3.2. If we form the generating function differently as $\beta(x) = \sum_{n \in \mathbb{Z}} \beta_n x^{-n-1}$, then we have

$$[\beta(x_1), \beta(x_2)] = \frac{1}{q - q^{-1}} \left(qx_1^{-2} \delta \left(\frac{qx_2}{x_1} \right) - q^{-1} x_1^{-2} \delta \left(\frac{x_2}{qx_1} \right) \right) c, \quad (3.5)$$

which does not look as neat as (3.3).

We say an H_q -module W is of *level* $\ell \in \mathbb{C}$ if c acts on W as scalar ℓ , and we say an H_q -module W is *restricted* if for any $w \in W$, $\beta_n w = 0$ for n sufficiently large.

Next we associate this Lie algebra with vertex algebras in terms of ϕ -coordinated quasi modules. We first construct a vertex algebra using another Lie algebra. Let B be a vector space over \mathbb{C} with basis $\{\beta^{(r)} \mid r \in \mathbb{Z}\}$. Equip B with a bilinear form defined by

$$\langle \beta^{(r)}, \beta^{(s)} \rangle = \frac{1}{q - q^{-1}} (\delta_{r,s+1} - \delta_{r,s-1}) = \frac{1}{q - q^{-1}} (\delta_{r,s+1} - \delta_{s,r+1}) \quad (3.6)$$

for $r, s \in \mathbb{Z}$. It can be readily seen that this bilinear form is skew-symmetric and non-degenerate. To the pair $(B, \langle \cdot, \cdot \rangle)$, we associate a Heisenberg Lie algebra

$$\hat{B} = B \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is central and

$$[a \otimes t^m, b \otimes t^n] = \langle a, b \rangle \delta_{m+n+1, 0} c \quad (3.7)$$

for $a, b \in B$, $m, n \in \mathbb{Z}$.

For $b \in B$, form a generating function

$$b(x) = \sum_{n \in \mathbb{Z}} b_n x^{-n-1}, \quad (3.8)$$

where $b_n = b \otimes t^n$. Now, the defining relations (3.7) amount to

$$[a(x_1), b(x_2)] = \langle a, b \rangle x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) c \quad (3.9)$$

for $a, b \in B$.

Set

$$\hat{B}_{\geq 0} = (B \otimes \mathbb{C}[t]) \oplus \mathbb{C}c, \quad \hat{B}_{< 0} = B \otimes t^{-1}\mathbb{C}[t^{-1}].$$

We see that $\hat{B}_{\geq 0}$ and $\hat{B}_{< 0}$ are Lie subalgebras and $\hat{B} = \hat{B}_{\geq 0} \oplus \hat{B}_{< 0}$ as a vector space. Let $\ell \in \mathbb{C}$ and denote by \mathbb{C}_ℓ the one-dimensional $\hat{B}_{\geq 0}$ -module with c acting as scalar ℓ and with $B \otimes \mathbb{C}[t]$ acting trivially. Form the induced module

$$V_{\hat{B}}(\ell, 0) = U(\hat{B}) \otimes_{U(\hat{B}_{\geq 0})} \mathbb{C}_\ell.$$

Set $\mathbf{1} = 1 \otimes 1 \in V_{\hat{B}}(\ell, 0)$. Identify B as a subspace of $V_{\hat{B}}(\ell, 0)$ through the linear map $b \in B \mapsto b(-1)\mathbf{1}$. It is now well known that there exists a vertex algebra structure on $V_{\hat{B}}(\ell, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and $Y(b, x) = b(x)$ for $b \in B$. Furthermore, B is a generating subspace of vertex algebra $V_{\hat{B}}(\ell, 0)$, where for $r, s \in \mathbb{Z}$ and for $n \geq 0$,

$$\beta_n^{(r)} \beta^{(s)} = \beta_n^{(r)} \beta_{-1}^{(s)} \mathbf{1} = \ell \langle \beta^{(r)}, \beta^{(s)} \rangle \delta_{n, 0} \mathbf{1} \quad (3.10)$$

as $\beta_n^{(r)} \mathbf{1} = 0$. It is clear that $V_{\hat{B}}(\ell, 0)$ is an irreducible \hat{B} -module. It follows that vertex algebra $V_{\hat{B}}(\ell, 0)$ is simple.

Furthermore we have:

Lemma 3.3. *The abelian group \mathbb{Z} acts on $V_{\hat{B}}(\ell, 0)$ as an automorphism group such that*

$$\rho_m(\beta^{(r)}) = \beta^{(r+m)} \quad \text{for } m, r \in \mathbb{Z}.$$

Proof. For $m \in \mathbb{Z}$, let $\hat{\rho}_m$ be the linear automorphism of B defined by $\hat{\rho}_m(\beta^{(n)}) = \beta^{(m+n)}$ for $n \in \mathbb{Z}$. It is clear that $\hat{\rho}_m$ preserves the bilinear form $\langle \cdot, \cdot \rangle$ (see (3.6)). It follows that $\hat{\rho}_m$ gives an automorphism of the Lie algebra H_q such that $\hat{\rho}_m(c) = c$ and

$$\hat{\rho}_m(\beta^{(n)} \otimes t^r) = \beta^{(m+n)} \otimes t^r \quad \text{for } n, r \in \mathbb{Z}.$$

This gives rise to an automorphism of $U(H_q)$, also denoted by $\hat{\rho}_m$. Furthermore, $\hat{\rho}_m$ gives rise to a linear automorphism ρ_m of $V_{\hat{B}}(\ell, 0)$ such that $\rho_m(\mathbf{1}) = \mathbf{1}$ and

$$\rho_m(av) = \hat{\rho}_m(a)\rho_m(v) \quad \text{for } a \in U(H_q), v \in V_{\hat{B}}(\ell, 0).$$

As $V_{\hat{B}}(\ell, 0)$ as a vertex algebra is generated by B , it follows that ρ_m is an automorphism of $V_{\hat{B}}(\ell, 0)$ viewed as a vertex algebra. This gives an action of \mathbb{Z} on $V_{\hat{B}}(\ell, 0)$ by automorphisms. \square

Set

$$\Gamma_q = \{q^n \mid n \in \mathbb{Z}\} \subset \mathbb{C}^\times.$$

Recall that a ϕ -coordinated quasi $V_{\hat{B}}(\ell, 0)$ -module (W, Y_W) is Γ_q -covariant if

$$Y_W(\rho_m(v), x) = Y_W(v, q^m x) \quad \text{for } v \in V_{\hat{B}}(\ell, 0), m \in \mathbb{Z}. \quad (3.11)$$

As our main result of this section we have:

Theorem 3.4. *Let W be a restricted H_q -module of level ℓ . Then there exists a Γ_q -covariant ϕ -coordinated quasi $V_{\hat{B}}(\ell, 0)$ -module structure on W , which is uniquely determined by $Y_W(\beta^{(r)}, x) = \beta(q^r x)$ for $r \in \mathbb{Z}$.*

Proof. Set $U_W = \{\beta(q^r x) \mid r \in \mathbb{Z}\} \subset \mathcal{E}(W)$. For $r, s \in \mathbb{Z}$, we have

$$[\beta(q^r x_1), \beta(q^s x_2)] = \frac{1}{q - q^{-1}} \left(\delta \left(q^{s-r+1} \frac{x_2}{x_1} \right) - \delta \left(q^{s-r-1} \frac{x_2}{x_1} \right) \right) \ell, \quad (3.12)$$

which implies

$$\left(\frac{x_1}{x_2} - q^{s-r+1} \right) \left(\frac{x_1}{x_2} - q^{s-r-1} \right) [\beta(q^r x_1), \beta(q^s x_2)] = 0. \quad (3.13)$$

Then U_W is a quasi local subset of $\mathcal{E}(W)$. By Corollary 2.12, U_W generates a vertex algebra V_W and W is a ϕ -coordinated quasi V_W -module where $Y_W(a(x), z) = a(z)$ for $a(x) \in V_W$.

With (3.13), by Lemma 6.7 of [Li6] we have

$$\begin{aligned} \beta(q^r x)_n^e \beta(q^s x) &= 0 \quad \text{for } r - s \neq \pm 1, n \geq 0, \\ \beta(q^r x)_n^e \beta(q^s x) &= 0 \quad \text{for } r - s = \pm 1, n \geq 1. \end{aligned}$$

If $r - s = 1$, also by Lemma 6.7 of [Li6] we have

$$\begin{aligned}
& (1 - q^{-2})\beta(q^r x_1)_0^e \beta(q^s x_2) \\
&= \text{Res}_{x_1} \frac{1}{x_1 - x} \left(\frac{x_1}{x} - 1 \right) \left(\frac{x_1}{x} - q^{-2} \right) \beta(q^r x_1) \beta(q^s x) \\
&\quad - \text{Res}_{x_1} \frac{1}{-x + x_1} \left(\frac{x_1}{x} - 1 \right) \left(\frac{x_1}{x} - q^{-2} \right) \beta(q^s x) \beta(q^r x_1) \\
&= \text{Res}_{x_1} x^{-2} (x_1 - q^{-2} x) [\beta(q^r x_1), \beta(q^s x)] \\
&= \frac{1}{q - q^{-1}} \text{Res}_{x_1} x^{-2} (x_1 - q^{-2} x) \left(\delta \left(\frac{x}{x_1} \right) - \delta \left(q^{-2} \frac{x}{x_1} \right) \right) \ell \\
&= \ell \frac{1 - q^{-2}}{q - q^{-1}}.
\end{aligned}$$

That is,

$$\beta(q^r x_1)_0^e \beta(q^s x_2) = \frac{\ell}{q - q^{-1}} 1_W.$$

Similarly, if $r - s = -1$, we have

$$\beta(q^r x_1)_0^e \beta(q^s x_2) = -\frac{\ell}{q - q^{-1}} 1_W.$$

Consequently, we have

$$[Y_{\mathcal{E}}^{\phi}(\beta(q^r x), x_1), Y_{\mathcal{E}}^{\phi}(\beta(q^s x), x_2)] = \frac{\ell}{q - q^{-1}} (\delta_{r,s+1} - \delta_{r,s-1}) x_1^{-1} \delta \left(\frac{x_2}{x_1} \right). \quad (3.14)$$

Thus V_W is a restricted \hat{B} -module of level ℓ with $\beta_n^{(r)} = \beta(q^r x)_n$ for $r, n \in \mathbb{Z}$. Furthermore, V_W together with vector 1_W is a vacuum \hat{B} -module of level ℓ . It follows from the construction of $V_{\hat{B}}(\ell, 0)$ that there exists a \hat{B} -module homomorphism ψ from $V_{\hat{B}}(\ell, 0)$ to V_W with $\psi(\mathbf{1}) = 1_W$. For $r \in \mathbb{Z}$, we have

$$\psi(\beta^{(r)}) = \psi(\beta_{-1}^{(r)} \mathbf{1}) = \beta(q^r x)_{-1} 1_W = \beta(q^r x) \in V_W.$$

It follows that ψ is a homomorphism of vertex algebras. Consequently, W is a ϕ -coordinated quasi $V_{\hat{B}}(\ell, 0)$ -module. For $m, r \in \mathbb{Z}$, we have

$$Y_W(\rho_m \beta^{(r)}, z) = Y_W(\beta^{(m+r)}, z) = \beta(q^{m+r} z) = Y_W(\beta^{(r)}, q^m z).$$

The covariance property follows from Lemma 2.14. \square

4 Heisenberg Lie algebra $\tilde{\mathcal{H}}_q$

In this section, we study another deformed Heisenberg Lie algebra. We explicitly construct quantum vertex algebras and we then associate these quantum vertex algebras and their ϕ -coordinated quasi modules to the deformed Heisenberg Lie algebra.

Definition 4.1. Let q be a complex number. Denote by $\tilde{\mathcal{H}}_q$ the Lie algebra with generators β_n ($n \in \mathbb{Z}$) and c , with c central, subject to relations

$$[\beta_m, \beta_n] = m(1 - q^{|m|})\delta_{m+n,0}c \quad (4.1)$$

for $m, n \in \mathbb{Z}$.

If $q = \pm 1$, $\tilde{\mathcal{H}}_q$ is just an abelian Lie algebra. If $q = 0$, this becomes the standard free field Heisenberg Lie algebra. For the rest of this section, we assume that q is neither zero nor a root of unity.

Form the generating function

$$\beta(x) = \sum_{n \in \mathbb{Z}} \beta_n x^{-n}. \quad (4.2)$$

Lemma 4.2. The defining relations (4.1) amount to

$$[\beta(x_1), \beta(x_2)] = \left(\left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{x_2}{x_1} \right) + \frac{qx_1/x_2}{(1 - qx_1/x_2)^2} - \frac{qx_2/x_1}{(1 - qx_2/x_1)^2} \right) c. \quad (4.3)$$

Furthermore, the latter is equivalent to

$$\begin{aligned} \beta(x_1)\beta(x_2) - \beta(x_2)\beta(x_1) &= \frac{qx_1/x_2}{(1 - qx_1/x_2)^2}c - \frac{q^{-1}x_1/x_2}{(1 - q^{-1}x_1/x_2)^2}c \\ &\quad + \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{x_2}{x_1} \right) - \delta \left(\frac{qx_2}{x_1} \right) \right) c. \end{aligned} \quad (4.4)$$

Proof. It is straightforward:

$$\begin{aligned} [\beta(x_1), \beta(x_2)] &= \sum_{m \in \mathbb{Z}} m(1 - q^{|m|})x_1^{-m}x_2^m c \\ &= \sum_{m \in \mathbb{Z}} mx_1^{-m}x_2^m c - \sum_{m \geq 0} mq^m x_1^{-m}x_2^m c - \sum_{m < 0} mq^{-m} x_1^{-m}x_2^m c \\ &= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\sum_{m \in \mathbb{Z}} x_1^{-m}x_2^m - \sum_{m \geq 0} q^m x_1^{-m}x_2^m - \sum_{m \leq 0} q^{-m} x_1^{-m}x_2^m \right) c \\ &= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\sum_{m \in \mathbb{Z}} x_1^{-m}x_2^m - \sum_{m \geq 0} q^m x_1^{-m}x_2^m - \sum_{m \geq 0} q^m x_1^m x_2^{-m} \right) c \\ &= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{x_2}{x_1} \right) - \frac{1}{1 - qx_2/x_1} - \frac{1}{1 - qx_1/x_2} \right) c \\ &= \left(\left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{x_2}{x_1} \right) - \frac{qx_2/x_1}{(1 - qx_2/x_1)^2} + \frac{qx_1/x_2}{(1 - qx_1/x_2)^2} \right) c. \end{aligned}$$

Using the calculation above we obtain

$$\begin{aligned}
& [\beta(x_1), \beta(x_2)] \\
&= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\sum_{m \in \mathbb{Z}} x_1^{-m} x_2^m - \sum_{m \geq 0} q^m x_1^{-m} x_2^m - \sum_{m \leq 0} q^{-m} x_1^{-m} x_2^m \right) c \\
&= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{x_2}{x_1} \right) - \delta \left(\frac{qx_2}{x_1} \right) + \sum_{m < 0} q^m x_1^{-m} x_2^m - \sum_{m \leq 0} q^{-m} x_1^{-m} x_2^m \right) c \\
&= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{x_2}{x_1} \right) - \delta \left(\frac{qx_2}{x_1} \right) + \frac{x_1/(qx_2)}{1 - x_1/(qx_2)} - \frac{1}{1 - qx_1/x_2} \right) c \\
&= \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{x_2}{x_1} \right) - \delta \left(\frac{qx_2}{x_1} \right) \right) c - \frac{q^{-1}x_1/x_2}{(1 - q^{-1}x_1/x_2)^2} c + \frac{qx_1/x_2}{(1 - qx_1/x_2)^2} c,
\end{aligned}$$

as desired. \square

From (4.4) we have

$$\begin{aligned}
& (x_1 - x_2)^2 (x_1 - qx_2)^2 \beta(x_1) \beta(x_2) \\
&= (x_1 - x_2)^2 (x_1 - qx_2)^2 \left(\beta(x_2) \beta(x_1) + \frac{qx_1/x_2}{(1 - qx_1/x_2)^2} c - \frac{q^{-1}x_1/x_2}{(1 - q^{-1}x_1/x_2)^2} c \right). \quad (4.5)
\end{aligned}$$

Definition 4.3. We say an $\tilde{\mathcal{H}}_q$ -module W is *restricted* if $\beta(x) \in \mathcal{E}(W)$, i.e., for any $w \in W$, $\beta_n w = 0$ for n sufficiently large. If c acts on W as a scalar $\ell \in \mathbb{C}$ we say W is of *level* ℓ .

Let W be a restricted $\tilde{\mathcal{H}}_q$ -module of level ℓ . Set

$$U_W = \{\beta(x), 1_W\} \subset \mathcal{E}(W).$$

From (4.5) we see that U_W is a quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$. By Theorem 2.11, U_W generates a weak quantum vertex algebra $\langle U_W \rangle_e$ with W as a ϕ -coordinated quasi module. Next, we determine this weak quantum vertex algebra.

Definition 4.4. Let \hat{B}_q denote the Lie algebra with generators $\beta_m^{(r)}$ ($r, m \in \mathbb{Z}$) and c , with c central, subject to relations

$$\begin{aligned}
& \beta^{(r)}(x_1) \beta^{(s)}(x_2) - \beta^{(s)}(x_2) \beta^{(r)}(x_1) \\
&= \iota_{x_2, x_1} \left(\frac{q^{r-s+1} e^{x_1 - x_2}}{(1 - q^{r-s+1} e^{x_1 - x_2})^2} - \frac{q^{r-s-1} e^{x_1 - x_2}}{(1 - q^{r-s-1} e^{x_1 - x_2})^2} \right) c \\
&+ (\delta_{r,s} - \delta_{r,s+1}) \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) c \quad (4.6)
\end{aligned}$$

for $r, s \in \mathbb{Z}$, where

$$\beta^{(r)}(x) = \sum_{m \in \mathbb{Z}} \beta_m^{(r)} x^{-m-1}. \quad (4.7)$$

Remark 4.5. For $n \in \mathbb{Z}$, set

$$f_n(x) = \iota_{x,0} \left(\frac{q^{n+1}e^x}{(1 - q^{n+1}e^x)^2} - \frac{q^{n-1}e^x}{(1 - q^{n-1}e^x)^2} \right) \in \mathbb{C}((x)), \quad (4.8)$$

where $\iota_{x,0}(\cdot)$ stands for the formal Laurent series expansion at $x = 0$. Then

$$\iota_{x_2,x_1} \left(\frac{q^{r-s+1}e^{x_1-x_2}}{(1 - q^{r-s+1}e^{x_1-x_2})^2} - \frac{q^{r-s-1}e^{x_1-x_2}}{(1 - q^{r-s-1}e^{x_1-x_2})^2} \right) = f_{r-s}(-x_2 + x_1)$$

for $r, s \in \mathbb{Z}$. We see that

$$f_n(x) = \delta_{n+1,0}x^{-2} - \delta_{n-1,0}x^{-2} + O(1). \quad (4.9)$$

Let \hat{B}_q^+ denote the subspace spanned by $\beta_n^{(r)}$ for $r \in \mathbb{Z}$, $n \geq 0$. Noticing that

$$\iota_{x_2,x_1} \left(\frac{q^{r-s+1}e^{x_1-x_2}}{(1 - q^{r-s+1}e^{x_1-x_2})^2} - \frac{q^{r-s-1}e^{x_1-x_2}}{(1 - q^{r-s-1}e^{x_1-x_2})^2} \right) = f_{r-s}(-x_2 + x_1)$$

involves only nonnegative powers of x_1 , we get

$$[\beta_m^{(r)}, \beta_n^{(s)}] = (\delta_{r,s} - \delta_{r,s+1})m\delta_{m+n,0}c \quad (4.10)$$

for $r, s, m, n \in \mathbb{Z}$ with $m \geq 0$. It follows that \hat{B}_q^+ is an abelian subalgebra.

A \hat{B}_q -module W is said to be *restricted* if $\beta^{(r)}(x) \in \mathcal{E}(W)$ for $r \in \mathbb{Z}$, i.e., for any $w \in W$, $r \in \mathbb{Z}$, $\beta_n^{(r)}w = 0$ for n sufficiently large. A \hat{B}_q -module W is said to be of *level* $\ell \in \mathbb{C}$ if c acts on W as scalar ℓ . A *vacuum* \hat{B}_q -module of level ℓ is a module W equipped with a vector $w_0 \in W$ such that W is cyclic on w_0 and

$$\beta_n^{(r)}w_0 = 0 \quad \text{for } r, n \in \mathbb{Z} \text{ with } n \geq 0.$$

Remark 4.6. Let (W, w_0) be a vacuum \hat{B}_q -module. By definition we have $W = U(\hat{B}_q)w_0$. On the other hand, it follows from (4.10) (and induction) that for any $r \in \mathbb{Z}$ and for any $u \in U(\hat{B}_q)$, $\beta_m^{(r)}u = u\beta_m^{(r)}$ for m sufficiently large. Then it follows that W is restricted.

Let $\ell \in \mathbb{C}$. Denote by $\mathbb{C}_\ell = \mathbb{C}$ the 1-dimensional $(\hat{B}_q^+ + \mathbb{C}c)$ -module with \hat{B}_q^+ acting trivially and with c acting as scalar ℓ . Form the induced module

$$V_{\hat{B}_q}(\ell, 0) = U(\hat{B}_q) \otimes_{U(\hat{B}_q^+ + \mathbb{C}c)} \mathbb{C}_\ell. \quad (4.11)$$

Set $\mathbf{1} = 1 \otimes 1 \in V_{\hat{B}_q}(\ell, 0)$. Identify $\beta^{(r)}$ with $\beta_{-1}^{(r)}\mathbf{1}$ for $r \in \mathbb{Z}$. It is clear that $V_{\hat{B}_q}(\ell, 0)$ is a vacuum \hat{B}_q -module of level ℓ , which is universal in the obvious sense.

Theorem 4.7. *Let $\ell \in \mathbb{C}$. There exists a weak quantum vertex algebra structure on $V_{\hat{B}_q}(\ell, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and $Y(\beta^{(r)}, x) = \beta^{(r)}(x)$ for $r \in \mathbb{Z}$. Furthermore, if $\ell \neq 0$, $V_{\hat{B}_q}(\ell, 0)$ is an irreducible quantum vertex algebra. On the other hand, for any restricted \hat{B}_q -module W of level ℓ , there exists a $V_{\hat{B}_q}(\ell, 0)$ -module structure uniquely determined by the condition that $Y_W(\beta^{(r)}, x) = \beta^{(r)}(x)$ for $r \in \mathbb{Z}$.*

Proof. Let W be any restricted \hat{B}_q -module of level ℓ . Set

$$U_W = \{1_W\} \cup \{\beta^{(r)}(x) \mid r \in \mathbb{Z}\}.$$

From the defining relations of \hat{B}_q , we see that U_W is an \mathcal{S} -local subset of $\mathcal{E}(W)$. By Theorem 2.7, U_W generates a weak quantum vertex algebra $\langle U_W \rangle$. Furthermore, it follows from Proposition 6.6 of [Li2] that

$$\begin{aligned} & Y_{\mathcal{E}}(\beta^{(r)}(x), x_1)Y_{\mathcal{E}}(\beta^{(s)}(x), x_2) - Y_{\mathcal{E}}(\beta^{(s)}(x), x_2)Y_{\mathcal{E}}(\beta^{(r)}(x), x_1) \\ = & \iota_{x_2, x_1} \left(\frac{q^{r-s+1}e^{x_1-x_2}}{(1 - q^{r-s+1}e^{x_1-x_2})^2} - \frac{q^{r-s-1}e^{x_1-x_2}}{(1 - q^{r-s-1}e^{x_1-x_2})^2} \right) \ell 1_W \\ & + \ell(\delta_{r,s} - \delta_{r,s+1}) \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) 1_W \end{aligned} \quad (4.12)$$

for $r, s \in \mathbb{Z}$. From this we see that $\langle U_W \rangle$ is a vacuum \hat{B}_q -module of level ℓ with $\beta^{(r)}(z)$ acting as $Y_{\mathcal{E}}(\beta^{(r)}(x), z)$ for $r \in \mathbb{Z}$. Then it follows from the universality of $V_{\hat{B}_q}(\ell, 0)$ that there exists a \hat{B}_q -module homomorphism ψ from $V_{\hat{B}_q}(\ell, 0)$ to $\langle U_W \rangle$, sending $\mathbf{1}$ to 1_W . For $r \in \mathbb{Z}$, we have

$$\psi(\beta_n^{(r)}v) = \beta^{(r)}(x)_n \psi(v) \quad \text{for } v \in V_{\hat{B}_q}(\ell, 0),$$

in particular,

$$\psi(\beta^{(r)}) = \psi(\beta_{-1}^{(r)}\mathbf{1}) = \beta^{(r)}(x)_{-1}1_W = \beta^{(r)}(x).$$

Specializing W to $V_{\hat{B}_q}(\ell, 0)$, by Theorem 2.9 of [Li3] with $U = \{\beta^{(r)} \mid r \in \mathbb{Z}\}$ and

$$Y_0(\beta^{(r)}, x) = \beta^{(r)}(x) \quad \text{for } r \in \mathbb{Z},$$

we have a weak quantum vertex algebra structure on $V_{\hat{B}_q}(\ell, 0)$ with all the requirements and such a structure is unique.

Come back to a general restricted \hat{B}_q -module W of level ℓ . We have a \hat{B}_q -module homomorphism ψ from $V_{\hat{B}_q}(\ell, 0)$ to $\langle U_W \rangle$, sending $\mathbf{1}$ to 1_W . For $r \in \mathbb{Z}$, we have

$$\psi(Y(\beta^{(r)}, z)v) = \psi(\beta^{(r)}(z)v) = Y_{\mathcal{E}}(\beta^{(r)}(x), z)\psi(v) = Y_{\mathcal{E}}(\psi(\beta^{(r)}), z)\psi(v)$$

for $v \in V_{\hat{B}_q}(\ell, 0)$. Since $\{\beta^{(r)} \mid r \in \mathbb{Z}\}$ generates $V_{\hat{B}_q}(\ell, 0)$, it follows that ψ is a homomorphism of weak quantum vertex algebras. As W is a canonical $\langle U_W \rangle$ -module, W becomes a $V_{\hat{B}_q}(\ell, 0)$ -module through the homomorphism ψ , where

$$Y_W(\beta^{(r)}, z) = Y_W(\psi(\beta^{(r)}), z) = Y_W(\beta^{(r)}(x), z) = \beta^{(r)}(z)$$

for $r \in \mathbb{Z}$. The uniqueness is clear.

Next, we prove that $V_{\hat{B}_q}(\ell, 0)$ is an irreducible \hat{B}_q -module. First we define a descending filtration for Lie algebra \hat{B}_q . For $n \geq 1$, set

$$\hat{B}_q[n] = \text{span}\{\beta_m^{(r)} \mid r, m \in \mathbb{Z}, m \geq n\},$$

and for $n \leq 0$, set

$$\hat{B}_q[n] = \text{span}\{\beta_m^{(r)} \mid r, m \in \mathbb{Z}, m \geq n\} + \mathbb{C}c.$$

Using (4.10) we get

$$[\hat{B}_q[m], \hat{B}_q[n]] \subset \hat{B}_q[m+n] \quad \text{for } m, n \in \mathbb{Z}.$$

Denote the associated graded Lie algebra by L . For $r, m \in \mathbb{Z}$, set

$$\bar{\beta}_m^{(r)} = \beta_m^{(r)} + \hat{B}_q[m+1] \in L$$

and set

$$\bar{c} = c + \hat{B}_q[1] \in L.$$

With (4.9) we see that L satisfies relations

$$\begin{aligned} & [\bar{\beta}^{(r)}(x_1), \bar{\beta}^{(s)}(x_2)] \\ &= (\delta_{r,s} - \delta_{r,s+1}) \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) \bar{c} + (\delta_{r-s+1,0} - \delta_{r-s-1,0}) (x_2 - x_1)^{-2} \bar{c} \\ &= \delta_{r,s} \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) \bar{c} + \delta_{r,s-1} \frac{1}{(x_2 - x_1)^2} \bar{c} - \delta_{r,s+1} \frac{1}{(x_1 - x_2)^2} \bar{c} \end{aligned} \quad (4.13)$$

for $r, s \in \mathbb{Z}$. In terms of components we have

$$[\bar{\beta}_m^{(r)}, \bar{\beta}_n^{(s)}] = \left(m\delta_{r,s} + \delta_{r,s-1} \frac{|m| - m}{2} - \delta_{r,s+1} \frac{|m| + m}{2} \right) \delta_{m+n,0} \bar{c} \quad (4.14)$$

for $m, n \in \mathbb{Z}$. In Lemma 4.8 below we shall prove that for nonzero $\ell \in \mathbb{C}$, every (nonzero) vacuum L -module of level ℓ is irreducible. Then by Proposition 2.11 of [KL], $V_{\hat{B}_q}(\ell, 0)$ is an irreducible \hat{B}_q -module. It follows that $V_{\hat{B}_q}(\ell, 0)$ as a (left) $V_{\hat{B}_q}(\ell, 0)$ -module is irreducible. Now, the proof is complete. \square

The following is the result we need at the end of the proof of Theorem 4.7:

Lemma 4.8. *Let L be a Heisenberg algebra with basis $\{c\} \cup \{\bar{\beta}_m^{(r)} \mid r, m \in \mathbb{Z}\}$, where c is central and*

$$[\bar{\beta}_m^{(r)}, \bar{\beta}_n^{(s)}] = \left(m\delta_{r,s} + \delta_{r,s-1} \frac{|m| - m}{2} - \delta_{r,s+1} \frac{|m| + m}{2} \right) \delta_{m+n,0} c$$

for $m, n, r, s \in \mathbb{Z}$. Suppose that W is an L -module of nonzero level ℓ equipped with a nonzero vector $w_0 \in V$, satisfying

$$W = U(L)w_0 \quad \text{and} \quad \bar{\beta}_m^{(r)} w_0 = 0 \quad \text{for } r, m \in \mathbb{Z} \text{ with } m \geq 0.$$

Then W is irreducible.

Proof. Set

$$V = \mathbb{C}[x_n^{(r)} \mid r \in \mathbb{Z}, n \geq 1],$$

a commutative polynomial algebra in variables $x_n^{(r)}$. For $r, n \in \mathbb{Z}$ with $n \geq 1$, set

$$\bar{\beta}_0^{(r)} = 0, \quad \bar{\beta}_{-n}^{(r)} = x_n^{(r)}, \quad \bar{\beta}_n^{(r)} = \ell n \left(\frac{\partial}{\partial x_n^{(r)}} - \frac{\partial}{\partial x_n^{(r-1)}} \right).$$

It is straightforward to check that this defines an L -module structure on V of level ℓ . Now we show that V is an irreducible L -module. Let A be a nonzero L -submodule of V . Set $\deg x_n^{(r)} = 1$ for $n, r \in \mathbb{Z}$ with $n \geq 1$. Let $P \in A$ be a nonzero polynomial with least degree. If $\deg P = 0$, then $A = V$ as V is clearly cyclic on 1. We next show that this must be the case. Assume $\deg P \geq 1$. There exists $r \in \mathbb{Z}$ such that

$$\frac{\partial P}{\partial x_m^{(s)}} = 0 \quad \text{for all } m \geq 1, s < r,$$

and $\frac{\partial P}{\partial x_n^{(r)}} \neq 0$ for some $n \geq 1$. Then

$$\bar{\beta}_n^{(r)} \cdot P = \ell n \left(\frac{\partial}{\partial x_n^{(r)}} - \frac{\partial}{\partial x_n^{(r-1)}} \right) P = \ell n \frac{\partial P}{\partial x_n^{(r)}}.$$

We have that $\frac{\partial P}{\partial x_n^{(r)}} \in A$ and $\frac{\partial P}{\partial x_n^{(r)}} \neq 0$, a contradiction. Therefore, V is irreducible. It follows from the standard argument (cf. [LL]) that W is isomorphic to V . \square

The following is analogous to Lemma 3.3:

Lemma 4.9. *For each $n \in \mathbb{Z}$, there exists an automorphism ρ_n of $V_{\hat{B}_q}(\ell, 0)$, which is uniquely determined by*

$$\rho_n(\beta^{(r)}) = \beta^{(n+r)} \quad \text{for } r \in \mathbb{Z}. \quad (4.15)$$

Furthermore, $\rho_0 = 1$ and $\rho_{m+n} = \rho_m \rho_n$ for $m, n \in \mathbb{Z}$.

Proof. First, for each given $n \in \mathbb{Z}$, from the defining relations of \hat{B}_q we see that there exists an automorphism σ_n of \hat{B}_q , which is uniquely determined by

$$\sigma_n(c) = c, \quad \sigma_n(\beta_m^{(r)}) = \beta_m^{(n+r)} \quad \text{for } r, m \in \mathbb{Z}.$$

Then σ_n induces an automorphism of $U(\hat{B}_q)$. Clearly, $\sigma_n(\hat{B}_q^+) = \hat{B}_q^+$. It follows that σ_n reduces to a linear automorphism ρ_n of $V_{\hat{B}_q}(\ell, 0)$ with $\rho_n(\mathbf{1}) = \mathbf{1}$ and

$$\rho_n(av) = \sigma_n(a)\rho_n(v) \quad \text{for } a \in U(\hat{B}_q), v \in V_{\hat{B}_q}(\ell, 0).$$

For $n, r \in \mathbb{Z}$, we have

$$\rho_n(\beta^{(r)}) = \rho_n(\beta_{-1}^{(r)}\mathbf{1}) = \sigma_n(\beta_{-1}^{(r)})\rho_n(\mathbf{1}) = \beta_{-1}^{(n+r)}\mathbf{1} = \beta^{(n+r)}.$$

$$\rho_n(Y(\beta^{(r)}, x)v) = \rho_n(\beta^{(r)}(x)v) = \beta^{(n+r)}(x)\rho_n(v) = Y(\rho_n(\beta^{(r)}), x)\rho_n(v)$$

for $v \in V_{\hat{B}_q}(\ell, 0)$. Consequently, ρ_n is an automorphism of weak quantum vertex algebra $V_{\hat{B}_q}(\ell, 0)$. \square

Next, we relate restricted $\widetilde{\mathcal{H}}_q$ -modules of level ℓ to ϕ -coordinated quasi $V_{\hat{B}_q}(\ell, 0)$ -modules. To achieve this goal we shall need the following result:

Lemma 4.10. *Let W be a vector space and let K be any $Y_{\mathcal{E}}^e$ -closed quasi compatible subspace of $\mathcal{E}(W)$ with $a(x), b(x) \in K$. Suppose that $a(x), b(x)$ satisfy relation*

$$[a(x_1), b(x_2)] = f(x_1/x_2) + \alpha \left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{x_2}{x_1} \right) + \beta \left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{qx_2}{x_1} \right), \quad (4.16)$$

where $f(x) \in \mathbb{C}(x)$, $\alpha, \beta, q \in \mathbb{C}$ with $q \neq 0, 1$. Then

$$[Y_{\mathcal{E}}^e(a(x), x_1), Y_{\mathcal{E}}^e(b(x), x_2)] = \iota_{x_2, x_1}(f(e^{x_1-x_2})) + \alpha \frac{\partial}{\partial x_2} x_1^{-1} \delta \left(\frac{x_2}{x_1} \right). \quad (4.17)$$

Proof. From the given relation we have

$$\begin{aligned} & (x_1 - x_2)^2 (x_1 - qx_2)^2 a(x_1) b(x_2) \\ &= (x_1 - x_2)^2 (x_1 - qx_2)^2 (b(x_2) a(x_1) + f(x_1/x_2)), \end{aligned} \quad (4.18)$$

which implies

$$(x_1 - x_2)^2 (x_1 - qx_2)^2 a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Then we have

$$\begin{aligned} & x^4 (e^z - 1)^2 (e^z - q)^2 Y_{\mathcal{E}}^e(a(x), z) b(x) \\ &= ((x_1 - x)^2 (x_1 - qx)^2 a(x_1) b(x)) \big|_{x_1 = xe^z} \\ &= \text{Res}_{x_1} x_1^{-1} \delta \left(\frac{xe^z}{x_1} \right) ((x_1 - x)^2 (x_1 - qx)^2 a(x_1) b(x)) \\ &= \text{Res}_{x_1} \frac{1}{x_1 - xe^z} ((x_1 - x)^2 (x_1 - qx)^2 a(x_1) b(x)) \\ &\quad - \text{Res}_{x_1} \frac{1}{-xe^z + x_1} ((x_1 - x)^2 (x_1 - qx)^2 a(x_1) b(x)) \\ &= \text{Res}_{x_1} \frac{1}{x_1 - xe^z} (x_1 - x)^2 (x_1 - qx)^2 a(x_1) b(x) \\ &\quad - \text{Res}_{x_1} \frac{1}{-xe^z + x_1} (x_1 - x)^2 (x_1 - qx)^2 (b(x) a(x_1) + f(x_1/x)). \end{aligned}$$

For convenience let us denote the last term by A . Noticing that A involves only nonnegative powers of z , we have

$$\begin{aligned} Y_{\mathcal{E}}^e(a(x), z) b(x) &= x^{-4} \iota_{z, 0} ((e^z - 1)^{-2} (e^z - q)^{-2}) A \\ &= x^{-4} \left(\frac{1}{(1 - q)^2} z^{-2} + \frac{q - 3}{(1 - q)^3} z^{-1} \right) A + O(z^0). \end{aligned} \quad (4.19)$$

As

$$\begin{aligned}\frac{1}{x_1 - xe^z}(x_1 - x)^2 &= (x_1 - x) + xz + O(z^2), \\ \frac{1}{-xe^z + x_1}(x_1 - x)^2 &= (x_1 - x) + xz + O(z^2), \\ (x_1 - qx_2)^2 \left(x_2 \frac{\partial}{\partial x_2} \right) \delta \left(\frac{qx_2}{x_1} \right) &= 0,\end{aligned}$$

we get

$$\begin{aligned}A &= \text{Res}_{x_1}(x_1 - x + xz)(x_1 - qx)^2 (a(x_1)b(x) - b(x)a(x_1) - f(x_1/x)) + O(z^2) \\ &= \alpha \text{Res}_{x_1}(x_1 - x + xz)(x_1 - qx)^2 \left(x \frac{\partial}{\partial x} \right) \delta \left(\frac{x}{x_1} \right) + O(z^2) \\ &= \alpha x^4 ((1 - q)^2 + z(1 - q)(3 - q)) + O(z^2).\end{aligned}$$

Combining this with (4.19) we obtain

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = \alpha z^{-2} + O(z^0),$$

which implies

$$a(x)_n^e b(x) = \delta_{n,1} \alpha 1_W \quad \text{for } n \geq 0. \quad (4.20)$$

With (4.18), by Proposition 5.3 of [Li6] (specially by (5.12) in the proof) we have

$$\begin{aligned}& Y_{\mathcal{E}}^e(a(x), x_1) Y_{\mathcal{E}}^e(b(x), x_2) - Y_{\mathcal{E}}^e(b(x), x_2) Y_{\mathcal{E}}^e(a(x), x_1) - \iota_{x_2, x_1} f(e^{x_1 - x_2})) \\ &= \sum_{n \geq 0} Y_{\mathcal{E}}^e(a(x)_n^e b(x), x_2) \frac{1}{n!} \left(\frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left(\frac{x_2}{x_1} \right).\end{aligned}$$

Then the desired relation follows from (4.20). \square

Recall $\Gamma_q = \{q^n \mid n \in \mathbb{Z}\} \subset \mathbb{C}^\times$.

Theorem 4.11. *Let W be a restricted $\tilde{\mathcal{H}}_q$ -module of level $\ell \in \mathbb{C}$. There exists a Γ_q -covariant ϕ -coordinated quasi module structure on W for $V_{\hat{B}_q}(\ell, 0)$, which is uniquely determined by the condition that $Y_W(\beta^{(r)}, x) = \beta(q^r x)$ for $r \in \mathbb{Z}$.*

Proof. Set

$$U_W = \{1_W\} \cup \{\beta(q^r x) \mid r \in \mathbb{Z}\} \subset \mathcal{E}(W).$$

Let $a, b \in \Gamma_q$. From (4.4) we have

$$\begin{aligned}& \beta(ax_1)\beta(bx_2) - \beta(bx_2)\beta(ax_1) \\ &= \ell \left(\frac{qab^{-1}x_1/x_2}{(1 - qab^{-1}x_1/x_2)^2} - \frac{q^{-1}ab^{-1}x_1/x_2}{(1 - q^{-1}ab^{-1}x_1/x_2)^2} \right) \\ & \quad + \ell \left(x_2 \frac{\partial}{\partial x_2} \right) \left(\delta \left(\frac{ba^{-1}x_2}{x_1} \right) - \delta \left(\frac{qba^{-1}x_2}{x_1} \right) \right).\end{aligned} \quad (4.21)$$

From this we see that U_W is a quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$. By Theorem 2.11, U_W generates a weak quantum vertex algebra $\langle U_W \rangle_e$ with W as a faithful ϕ -coordinated quasi module.

With (4.21), by Lemma 4.10 we have

$$\begin{aligned} & Y_{\mathcal{E}}^e(\beta(ax), x_1)Y_{\mathcal{E}}^e(\beta(bx), x_2) - Y_{\mathcal{E}}^e(\beta(bx), x_2)Y_{\mathcal{E}}^e(\beta(ax), x_1) \\ = & \iota_{x_2, x_1} \left(\frac{qab^{-1}e^{x_1-x_2}}{(1-qab^{-1}e^{x_1-x_2})^2} - \frac{q^{-1}ab^{-1}e^{x_1-x_2}}{(1-q^{-1}ab^{-1}e^{x_1-x_2})^2} \right) \ell \\ & + \ell(\delta_{a,b} - \delta_{a,qb}) \left(\frac{\partial}{\partial x_2} \right) x_1^{-1} \delta \left(\frac{x_2}{x_1} \right). \end{aligned} \quad (4.22)$$

It follows that $\langle U_W \rangle_e$ is a restricted \hat{B}_q -module of level ℓ with $\beta^{(r)}(z)$ acting as $Y_{\mathcal{E}}^e(\beta(q^r x), z)$ for $r \in \mathbb{Z}$. We also have $Y_{\mathcal{E}}^e(\beta(q^r x), z)1_W \in \langle U_W \rangle_e[[z]]$ for $r \in \mathbb{Z}$. Then $\langle U_W \rangle_e$ together with 1_W is a vacuum \hat{B}_q -module of level ℓ . By the universality of $V_{\hat{B}_q}(\ell, 0)$, there exists a \hat{B}_q -module homomorphism ψ from $V_{\hat{B}_q}(\ell, 0)$ to $\langle U_W \rangle_e$ with $\psi(\mathbf{1}) = 1_W$. For $r \in \mathbb{Z}$, we have

$$\psi(\beta^{(r)}) = \psi(\beta_{-1}^{(r)}\mathbf{1}) = \beta(q^r x)_-^e 1_W = \beta(q^r x).$$

Furthermore, we have

$$\psi(Y(\beta^{(r)}, z)v) = \psi(\beta^{(r)}(z)v) = Y_{\mathcal{E}}^e(\beta(q^r x), z)\psi(v) = Y_{\mathcal{E}}^e(\psi(\beta^{(r)}), z)\psi(v)$$

for $r \in \mathbb{Z}$, $v \in V_{\hat{B}_q}(\ell, 0)$. Since $V_{\hat{B}_q}(\ell, 0)$ as a weak quantum vertex algebra is generated by elements $\beta^{(r)}$ ($r \in \mathbb{Z}$), it follows that ψ is a homomorphism of weak quantum vertex algebras. Consequently, W is a ϕ -coordinated quasi $V_{\hat{B}_q}(\ell, 0)$ -module where for $r \in \mathbb{Z}$,

$$Y_W(\beta^{(r)}, z) = Y_W(\psi(\beta^{(r)}), z) = Y_W(\beta(q^r x), z) = \beta(q^r z).$$

For $n, r \in \mathbb{Z}$, we have

$$Y_W(\rho_n(\beta^{(r)}), x) = Y_W(\beta^{(n+r)}, x) = \beta(q^{n+r}x) = Y_W(\beta^{(r)}, q^n x).$$

Since $V_{\hat{B}_q}(\ell, 0)$ is generated by elements $\beta^{(r)}$ for $r \in \mathbb{Z}$, the Γ_q -covariance follows from Lemma 2.14. Therefore, W is a Γ_q -covariant ϕ -coordinated quasi $V_{\hat{B}_q}(\ell, 0)$ -module. \square

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